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On sound generation by weakly nonlinear interactions of surface gravity waves

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The theory of sound generation by weakly nonlinearly interacting ocean surface waves is examined. The main conclusion is that this mechanism may not be a strong generator of ocean sound. It is shown that at low frequency and with small wind the sound generated by this mechanism is weak in comparison with that directly radiated by the turbulent airflow, the flow which is also the cause of surface waves. With increasing frequency and/or wind speed, the sound from surface-wave interactions becomes appreciable, but it is found that the condition for this sound overwhelming the aerial turbulence radiation implies the precise condition at which fully nonlinear surface motions occur. In that case processes such as splashing of water sprays by breaking waves become the main cause of ocean noise. In fact it seems that the weakly nonlinear mechanism proposed by Brekhovskikh is never an important source of sound in the real ocean.

1. Introduction

This paper concerns the theoretical modelling of sound generation processes in the ocean. The problem is formulated as follows. The two fluids, air and water, are separated by an infinite interface above which there is a turbulent airflow. The airflow generates sound directly and also produces surface gravity waves. We seek to determine the influence of weakly nonlinear interactions of that surface wave field on the sound generation process. The basic equations, derived by following Lighthill's (1952) procedure, are solved by making use of the method developed by Ffowcs Williams & Hawkings (1969) and Dowling, Ffowcs Williams & Goldstein (1978). We show that the underwater sound is caused by direct turbulence radiation, together with a surface-induced sound which, in the weakly nonlinear model, is proportional to the squared surface displacement.

The sound from surface waves is quantitatively examined in §3. We derive a formula that relates the spectrum of sound pressure in deep water to the surface wave spectrum. Our result is in agreement with the Brekhovskikh (1966) theory, which ascribes background oceanic noise to weakly nonlinear interactions of ocean surface waves. Despite the progress achieved during the past two decades in developing and utilizing the Brekhovskikh theory to explain the mechanism of ocean noise generation, agreement between theory and experiments is still not completely satisfactory. In view of this and considering what seems to us a questionable 'weak interaction' presumption in the theory, a new look is needed to consider carefully whether this interaction mechanism is really responsible for the ocean noise. That is what has motivated this study.

An obvious weak point in the wave–wave interaction theory is the neglect of the airflow that drives the waves. Urick (1967) has emphasized this point in the past and our theoretical treatment that follows is effectively a quantitative restatement of Urick's view. The neglect of the airflow may not be adequate, because it also neglects a sound field radiated by that flow. This aerial turbulence radiation may sometimes be dominant over the surface-induced sound. To decide the relative acoustic radiation efficiencies of turbulence sources and surface waves, we calculate and compare the respective sound powers from these two kinds of sources. The incorporation of gravity into the Lighthill acoustic analogy enables us to examine analytically the surface wave field produced by the turbulent airflow. We suppose that the airflow is of finite extent so that Olbers' paradox is avoided (for the concept of 'Olbers' paradox', see Ffowcs Williams 1982). In this way, our theory unambiguously includes the necessary basis for establishing the relative importance of the turbulence radiation and the sound induced by surface motions. We find that at frequency ω the ratio of direct turbulence radiation to the sound field generated indirectly by the turbulence-induced surface waves is of the order $10^3 \times (g^6/Lc_a u^4 \omega^7)$ or $10^6 \times (g^4/Lc_a u^2 \omega^5)$, corresponding respectively to the cases of ω smaller and bigger than g/U. L is the linear dimension of the turbulence source region, g the gravitational acceleration, c_a the constant sound speed in air, and u and U respectively denote the r.m.s. turbulence velocity and the uniform wind speed. These results indicate that at low frequency and small wind speed, ocean noise probably arises directly from the turbulent airflow and not from the nonlinear interactions of surface waves.

As frequency and/or wind speed increases, surface-induced sound becomes appreciable, because nonlinear effects in the surface deformation then become increasingly important. However the increase in wind speed soon causes surface waves to depart from the regime of 'weak interaction'. The Brekhovskikh theory is strictly applicable to situations where the ocean surface is continuous and single-valued, and the surface wave slope is much less than unity. High wind causes violent surface agitations that inevitably invalidate these two assumptions. We will deduce that the surface wave interactions can be important, compared with the direct aerial turbulence radiation, only at frequencies much greater than $15.4(g^4/Lc_a u^2)^{\frac{1}{5}}$. But this requirement always implies the precise condition for surface waves to be in a fully nonlinear state where they break, that is, $\omega \gg 2.2(g/u_*)(u_*^2/gL)^{\frac{1}{4}}$, u_* being the friction velocity (Phillips 1977). In this situation the Brekhovskikh theory fails. Noisy processes such as splashing of water sprays on the surface then become the dominant cause of underwater sound. This leads us to conclude that, in the natural ocean, weakly nonlinear interactions between surface waves are probably not significant sources of underwater sound.

2. Formulation and solution of the hydroacoustic problem

We consider a turbulent airflow over the ocean surface whose undisturbed position coincides with the plane $y_3 = 0$ of a Cartesian-coordinate system $\boldsymbol{y}$. We incorporate gravitational effects and ignore viscosity in writing the appropriate form of Lighthill's (1952) acoustic analogy in the airflow as

$$\left(\frac{\partial^2}{\partial \tau^2} - c_a^2 \nabla^2 - g\frac{\partial}{\partial y_3}\right)\rho_a' = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}, \tag{2.1}$$

where

$$T_{ij} = \rho u_i u_j + \delta_{ij}(p_a' - c_a^2 \rho_a')$$

is the Lighthill stress tensor. The subscript a is used to denote quantities in air. Pressure and density perturbations are over the mean values $\bar{p}(y_3)$ and $\bar{\rho}(y_3)$ which are determined by hydrostatic relations, and g is the value of the gravitational acceleration.

The governing equation in water can be similarly written with the subscript w symbolizing quantities in water. We restrict ourselves to the situation where motions in the water body are essentially linear, so that the Lighthill stress tensor there vanishes and the basic equation is

$$\left(\frac{\partial^2}{\partial\tau^2}-c_{\mathrm{w}}^2\nabla^2-g\frac{\partial}{\partial y_3}\right)\rho'_{\mathrm{w}}=0. \qquad (2.2)$$

We ignore the Lighthill stress tensor in the water body because it can only induce very weak acoustic sources. If the surface waves are only weakly nonlinear, the quadrupoles in the water body, as well as the dipoles on the water surface which, as will be seen shortly, result from the inclusion of weak nonlinearity in the boundary conditions at the surface, have a strength proportional to the square of that wave field. Considering that the quadrupoles occupy a layer of thickness of the same order as a typical surface wavelength, it is evident that the quadrupoles are less efficient than the dipoles by the typical Mach number of the water elements times the typical surface wavelength divided by the much larger acoustic wavelength. Hence the quadrupoles can be discarded. Now we define a generalized function $f(\boldsymbol{y},\tau)$ in such a way that it is positive in air and negative in water. The air–water interface can then be expressed as $f(\boldsymbol{y},\tau)=0$ and the kinematic and dynamic conditions on the interface can be written as

$$\frac{\mathrm{D}f(\boldsymbol{y},\tau)}{\mathrm{D}\tau}=0,\quad p_{\mathrm{a}}=p_{\mathrm{w}}\quad\text{on}\quad f(\boldsymbol{y},\tau)=0. \qquad (2.3)$$

Equations (2.1), (2.2) and (2.3), together with conditions at infinity, formulate the problem uniquely. To solve it we follow the method developed by Ffowcs Williams & Hawkings (1969) and Dowling *et al.* (1978), and introduce a generalized function $H\{f(\boldsymbol{y},\tau)\}$, equal to one in air and zero in water. We multiply (2.1) by H, transfer it through the differential operators and re-arrange terms by using the equations of motion and the condition $\mathrm{D}f/\mathrm{D}\tau=0$. As a result, an equation can be derived for the generalized function $H\rho'_{\mathrm{a}}$,

$$\left(\frac{\partial^2}{\partial\tau^2}-c_{\mathrm{a}}^2\nabla^2-g\frac{\partial}{\partial y_3}\right)H\rho'_{\mathrm{a}}=\frac{\partial^2 HT_{ij}}{\partial y_i\partial y_j}-\frac{\partial}{\partial\tau}\left(\bar{\rho}_{\mathrm{a}}\frac{\partial H}{\partial\tau}\right)-\frac{\partial}{\partial y_i}\left(p'_{\mathrm{a}}\frac{\partial H}{\partial y_i}\right).$$

This is essentially the Ffowcs Williams–Hawkings equation except for the extra term proportional to g, which accounts for gravitational effects. It has been deliberately retained because of its importance in the production of surface waves.

We choose to work with the Green function $G(\boldsymbol{y},\tau/\boldsymbol{x},t)$ defined by the adjoint form of (2.1) with the source term vanishing, that is,

$$\left(\frac{\partial^2}{\partial\tau^2}-c_{\mathrm{a}}^2\nabla^2+g\frac{\partial}{\partial y_3}\right)G(\boldsymbol{y},\tau/\boldsymbol{x},t)=0, \qquad (2.4)$$

where G is assumed to be incoming at (y_α,τ) infinity and $y_3\rightarrow+\infty$. Multiplying this equation by $H\rho'_{\mathrm{a}}$ and integrating it over the whole $(\boldsymbol{y},\tau)$-space, we find after some partial integrations

$$0=\int_\infty\left(HT_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}+\bar{\rho}_{\mathrm{a}}\frac{\partial H}{\partial\tau}\frac{\partial G}{\partial\tau}+p'_{\mathrm{a}}\frac{\partial H}{\partial y_i}\frac{\partial G}{\partial y_i}\right)\mathrm{d}^3y\,\mathrm{d}\tau. \qquad (2.5)$$

The foregoing procedure also applies to (2.2) which governs motions in water if it is multiplied by a generalized function defined by $(1-H)$. The similar result is then found to be

$$(1-H)\,\rho_w'(\boldsymbol{x},t) = \int_\infty \left(-\bar{\rho}_w \frac{\partial H}{\partial \tau}\frac{\partial \bar{G}}{\partial \tau} - p_w' \frac{\partial H}{\partial y_i}\frac{\partial \bar{G}}{\partial y_i}\right) \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau, \tag{2.6}$$

where $\bar{G}(\boldsymbol{y},\tau/\boldsymbol{x},t)$ is defined by the equation

$$\left(\frac{\partial^2}{\partial \tau^2} - c_w^2 \nabla^2 + g\frac{\partial}{\partial y_3}\right)\bar{G}(\boldsymbol{y},\tau/\boldsymbol{x},t) = \delta(\boldsymbol{y}-\boldsymbol{x},\tau-t), \tag{2.7}$$

with incoming behaviour at (y_α, τ) infinity and $y_3 \rightarrow -\infty$. The addition of (2.5) and (2.6) immediately gives the density fluctuation in the water,

$$\rho_w'(\boldsymbol{x},t) = \int_\infty \left[HT_{ij}\frac{\partial^2 G}{\partial y_i \partial y_j} + \left(\bar{\rho}_a \frac{\partial G}{\partial \tau} - \bar{\rho}_w \frac{\partial \bar{G}}{\partial \tau}\right)\frac{\partial H}{\partial \tau} + \left(p_a' \frac{\partial G}{\partial y_i} - p_w' \frac{\partial \bar{G}}{\partial y_i}\right)\frac{\partial H}{\partial y_i}\right] \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau.$$

The last two terms, containing derivatives of H, can be transferred into surface integrals by making use of properties of generalized functions (see Ffowcs Williams & Hawkings), which yields

$$\rho_w'(\boldsymbol{x},t) = \int_\infty HT_{ij}(\boldsymbol{y},\tau)\frac{\partial^2 G}{\partial y_i \partial y_j}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau + \int_\tau \int_s \left[\left(\bar{\rho}_a \frac{\partial G}{\partial \tau} - \bar{\rho}_w \frac{\partial G}{\partial \tau}\right)\frac{\partial f}{\partial \tau} + \left(p_a' \frac{\partial G}{\partial y_i} - p_w' \frac{\partial \bar{G}}{\partial y_i}\right)\frac{\partial f}{\partial y_i}\right]\frac{\mathrm{d}s}{|\boldsymbol{\nabla} f|}\mathrm{d}\tau$$

where $\mathrm{d}s$ is the surface element on $f(\boldsymbol{y},\tau) = 0$.

To simplify this result, we use the condition $p_a = p_w$ on $f(\boldsymbol{y},\tau) = 0$ and regard the air–water interface as being slightly disturbed from its initial position $y_3 = 0$ so that $f(\boldsymbol{y},\tau) = y_3 - \zeta(y_\alpha,\tau)$ is continuous and single-valued, ζ being the surface displacement and y_α the horizontal coordinates. On this account, the surface integrals can be projected from the curved surface $y_3 = \zeta$ onto $y_3 = 0$ by noticing that $\mathrm{d}s/|\boldsymbol{\nabla} f| = \mathrm{d}^2 y_\alpha$. Hence we have

$$\rho_w'(\boldsymbol{x},t) = \int_\infty HT_{ij}\frac{\partial^2 G}{\partial y_i \partial y_j}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau + \int_\tau \int_{y_\alpha} F(y_\alpha, y_3 = \zeta, \tau)\,\mathrm{d}^2 y_\alpha\,\mathrm{d}\tau,$$

where the surface integration is to be performed on the plane $y_3 = 0$ but its integrand F is calculated at $y_3 = \zeta$, F being given by

$$F(y_\alpha,\zeta,\tau) = p_w'\left(\frac{\partial G}{\partial y_3} - \frac{\partial \bar{G}}{\partial y_3}\right) + (\bar{p}_w - \bar{p}_a)\left(\frac{\partial G}{\partial y_3} - \frac{\partial \zeta}{\partial y_\alpha}\frac{\partial G}{\partial y_\alpha}\right) - \frac{\partial \zeta}{\partial \tau}\left(\bar{\rho}_a \frac{\partial G}{\partial \tau} - \bar{\rho}_w \frac{\partial G}{\partial \tau}\right) - p_w' \frac{\partial \zeta}{\partial y_\alpha}\left(\frac{\partial G}{\partial y_\alpha} - \frac{\partial \bar{G}}{\partial y_\alpha}\right).$$

(Here the overbar above p and ρ denotes mean values that depend on y_3.) This function can be expanded from $y_3 = \zeta$ to $y_3 = 0$ in a Taylor series. The convergence of this expansion is guaranteed provided that the surface wave slope is much less than unity. Carrying out this expansion and integrating the results by parts, it is found that

$$\int_\tau \int_{y_\alpha} F(y_\alpha, y_3 = \zeta, \tau)\,\mathrm{d}^2 y_\alpha\,\mathrm{d}\tau = \int_\tau \int_{y_\alpha} F_1(y_\alpha, y_3 = 0, \tau)\,\mathrm{d}^2 y_\alpha\,\mathrm{d}\tau$$

where

$$F_1(y_\alpha, 0, \tau) = p'_w\left(\frac{\partial G}{\partial y_3} - \frac{\partial \overline{G}}{\partial y_3}\right) + \zeta\left[\rho_a\left(\frac{\partial^2 G}{\partial \tau^2} + g\frac{\partial G}{\partial y_3}\right) - \rho_w\left(\frac{\partial^2 \overline{G}}{\partial \tau^2} + g\frac{\partial \overline{G}}{\partial y_3}\right)\right]$$
$$+\zeta p'_w \nabla^2(G-\overline{G}) + \zeta\frac{\partial p'_w}{\partial y_\alpha}\frac{\partial(G-\overline{G})}{\partial y_\alpha} - \tfrac{1}{2}\rho_w \zeta^2 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial G}{\partial y_3}\right),$$

with ρ_a and ρ_w respectively denoting the mean densities of air and water on the undisturbed interface. In deriving this result, the Taylor expansion of F has been truncated after the term proportional to ζ^2. This is justified by the fact that the ignored terms are smaller than the remainder at least by a factor equal to the surface wave slope that is much less than unity. Terms explicitly smaller by the factor $g/\omega c_w$ which is less than 0.01 for frequencies above 0.1 Hz, ω being the angular frequency, or by the ratio $\rho_a/\rho_w \approx 10^{-3}$, have also been neglected.

In F_1, the first two terms are linear in the perturbation variable. They often pose a difficult task in interpreting the final solution so that it is appropriate to eliminate them by imposing conditions on G and $\overline{G}$. Obviously, it is required that

$$\frac{\partial G}{\partial y_3} = \frac{\partial \overline{G}}{\partial y_3},$$

and
$$\rho_a\left(\frac{\partial^2 G}{\partial \tau^2} + g\frac{\partial G}{\partial y_3}\right) = \rho_w\left(\frac{\partial^2 \overline{G}}{\partial \tau^2} + g\frac{\partial \overline{G}}{\partial y_3}\right),$$

on $y_3 = 0$. The Green functions G and $\overline{G}$ are also uniquely determined by these two conditions, together with their defining equations (2.4) and (2.7). They can be found by Fourier transformation. The derivation is straightforward, but tedious in algebra, so that we will not give the details here but simply claim that we find

$$G = \frac{1}{(2\pi)^3 c_w^2}\int_\infty \frac{\rho_w \omega^2}{F_d(k_\alpha, \omega)} \exp\left(\frac{g}{2c_a^2}y_3 - \frac{g}{2c_w^2}x_3\right) e^{i[\gamma_a y_3 - \gamma_w x_3]} e^{-i[k_\alpha(y_\alpha - x_\alpha) + \omega(\tau - t)]} \, d\omega \, d^2k_\alpha$$

and

$$\overline{G} = \frac{1}{(2\pi)^3 c_w^2}\int_\infty \frac{1}{2i\gamma_w}\left(\frac{F_n(k_\alpha, \omega)}{F_d(k_\alpha, \omega)} e^{-i\gamma_w(y_3 + x_3)} - e^{i\gamma_w|y_3 - x_3|}\right)$$
$$\times \exp\left(\frac{g}{2c_w^2}(y_3 - x_3)\right) e^{-i[k_\alpha(y_\alpha - x_\alpha) + \omega(\tau - t)]} \, d\omega \, d^2k_\alpha$$

where k_α and ω are, respectively, the two-dimensional wavenumber and the angular frequency; $\gamma^2 = (\omega/c)^2 - k_\alpha^2 - (g/2c^2)^2$ with appropriate subscripts. The roots of γ are determined by the incoming behaviour of the two Green functions; when real they have the opposite sign to ω, while when purely imaginary $\mathrm{Im}(\gamma_a)$ and $\mathrm{Im}(\gamma_w)$ are always positive. F_n and F_d are given by

$$F_n(k_\alpha, \omega) = \rho_w\left(\frac{g}{2c_a^2} + i\gamma_a\right)\left(g\left(\frac{g}{2c_w^2} + i\gamma_w\right) - \omega^2\right) - \rho_a\left(\frac{g}{2c_w^2} + i\gamma_w\right)\left(g\left(\frac{g}{2c_a^2} + i\gamma_a\right) - \omega^2\right),$$

and

$$F_d(k_\alpha, \omega) = \rho_w\left(\frac{g}{2c_a^2} + i\gamma_a\right)\left(g\left(\frac{g}{2c_w^2} - i\gamma_w\right) - \omega^2\right) - \rho_a\left(\frac{g}{2c_w^2} - i\gamma_w\right)\left(g\left(\frac{g}{2c_a^2} + i\gamma_a\right) - \omega^2\right).$$

With this determination of G and $\overline{G}$, the linear terms in F_1 all vanish. Hence we have

$$F_1(y_\alpha, 0, \tau) = \zeta p'_{\mathrm{w}} \nabla^2(G-\overline{G}) + \zeta \frac{\partial p'_{\mathrm{w}}}{\partial y_\alpha} \frac{\partial (G-\overline{G})}{\partial y_\alpha} - \tfrac{1}{2}\rho_{\mathrm{w}} \zeta^2 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial G}{\partial y_3}\right), \tag{2.8}$$

where all quantities are evaluated at $y_3 = 0$. F_1 can be simplified still further by careful dimensional analysis, which reveals that the last term is the leading term. This becomes apparent once we scale the pressure fluctuation p'_{w} on the mean position of the water surface as of order $\rho_{\mathrm{w}} g \zeta$, which results from the momentum equation, as will be seen at the beginning of §4. The first nonlinear term is then smaller than the leading term by the ratio of the surface wavelength to the acoustic wavelength. The second is negligibly small because

$$\zeta \frac{\partial p'_{\mathrm{w}}}{\partial y_\alpha} = \frac{\partial}{\partial y_\alpha}(\tfrac{1}{2}\rho_{\mathrm{w}} g \zeta^2). \tag{2.9}$$

Integrating the second term of (2.8) by parts and making a transfer of the y_α-derivative onto the Green functions, it can be seen that it is smaller than the leading term by the small factor $g/\omega c_{\mathrm{w}}$.

In examining the effect on sound generation of nonlinear deformations of material surfaces, Howe (1985) derives a result similar to (2.8), but he identifies the second term as dominant. This is not the case in our ocean-sound problem. The ocean depth in our problem is assumed infinite (or much bigger than the acoustic wavelength), which is different from Howe's problem where the fluid under the turbulent flow is a thin layer with a rigid lower boundary. By assuming a depth h for our ocean problem, the foregoing analysis can be similarly carried out and it can be shown that the ratio of the third to the second term in (2.8), letting gravity vanish (to be consistent with Howe's problem), is of the order $\sin(h\omega/c_{\mathrm{w}})$. In Howe's case, he supposes that $h\omega/c_{\mathrm{w}} \ll 1$ so that the second term is dominant. But in our problem, $h\omega/c_{\mathrm{w}}$ is very much larger than unity. Hence these two terms are of the same order even in this gravity-free situation. Furthermore, since we are considering gravity waves, p'_{w} in our problem can be definitely scaled on $\rho_{\mathrm{w}} g\zeta$, which allows the second term in (2.8) to be converted into a divergence form as shown by (2.9), that degenerates further to a higher order. The third term can then be identified as the dominant term. Hence the solution can eventually be written as

$$\rho'_{\mathrm{w}}(\boldsymbol{x}, t) = \rho_{\mathrm{t}}(\boldsymbol{x}, t) + \rho_{\mathrm{s}}(\boldsymbol{x}, t)$$

where

$$\rho_{\mathrm{t}}(\boldsymbol{x}, t) = \int_\infty HT_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \mathrm{d}^3\boldsymbol{y}\, \mathrm{d}\tau, \tag{2.10}$$

and

$$\rho_{\mathrm{s}}(\boldsymbol{x}, t) = -\tfrac{1}{2}\rho_{\mathrm{w}} \int_\tau \int_{y_\alpha} \zeta^2(y_\alpha, \tau) \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial G}{\partial y_3}\right) \mathrm{d}^2 y_\alpha\, \mathrm{d}\tau. \tag{2.11}$$

Here ρ_{t} is the density fluctuation caused by the turbulent airflow and ρ_{s} is that induced by surface waves, which, in the weakly nonlinear model, is an integral on the mean position of the ocean surface and involves only the square of the surface elevation.

3. Sound from surface waves in the Brekhovskikh scheme

The theory of sound generation by weakly nonlinear interactions of ocean surface waves was first derived by Brekhovskikh (1966). Some others (e.g. Hughes 1976) have used slightly different methods to approach the problem, but the basic mathematical procedures and their main results are the same. A small parameter perturbation is used to convert the hydrodynamic equations to a series of linear equations of different orders, and a Taylor expansion transfers the boundary conditions from the moving surface to its mean position. When only second-order equations are considered (the weakly nonlinear assumption), this theory relates sound to a surface wave field in a quadratic way. This Brekhovskikh procedure is different from that of the previous section where, by making use of an appropriate form of the Lighthill acoustic analogy, we have also derived an expression for the surface-induced sound in terms of the squared surface displacement, but the surface term is no longer alone. Since we will discuss the relative importance of the weakly nonlinear interaction mechanism by making use of our results, it is appropriate to first establish the identity of our surface-induced sound to that in the Brekhovskikh theory.

In the previous section, we have derived the wave field in the ocean under a turbulent airflow in terms of the Green function G. At this stage, the effect of gravity can be made clear by examining this Green function. For every real k_α and ω, the integrand of G has a singularity (F_d has a zero) which is found near $g|k_\alpha| = \omega^2$. Obviously this singularity gives rise to gravity waves. Hence the turbulent airflow not only radiates sound but also induces ocean surface waves, both of which are contained in our theory which therefore should include the basis for establishing their relative importance. Apart from this introduction of surface waves, gravity does not seem to be important. This simplifies the problem considerably; terms explicitly proportional to g can be omitted in the sound field provided that we bear in mind the implication of the existence of gravity surface waves. On this account, the sound pressure caused by surface motions can be written from (2.11) as

$$p_s(\boldsymbol{x}, t) = \frac{i\rho_w^2}{2(2\pi)^3} \int_\infty \frac{\zeta^2 \omega^4 \gamma_a}{F_d(k_\alpha, \omega)} e^{-i[\gamma_w x_3 + k_\alpha(y_\alpha - x_\alpha) + \omega(\tau - t)]} d^2k_\alpha \, d^2y_\alpha \, d\omega \, d\tau.$$

The autocorrelation function of this pressure field can be obtained by multiplying it by itself calculated at $t+t'$, t' being the time separation, and averaging the result. Then a Fourier transform with respect to t' gives the frequency spectrum, which we denote by $P_s(\boldsymbol{x}, \omega)$,

$$P_s(\boldsymbol{x}, \omega) = \frac{\rho_w^4 \omega^8}{4(2\pi)^2} \int_\infty \frac{|\gamma_a|^2}{|F_d(k_\alpha, \omega)|^2} e^{i(\gamma_w^* - \gamma_w) x_3} N(y'_\alpha, \tau') \, e^{i(k_\alpha y'_\alpha + \omega\tau')} d^2k_\alpha \, d^2y'_\alpha \, d\tau', \quad (3.1)$$

where γ_w^* is the complex conjugate of γ_w; $|z|$ is the modulus of z and N is a fourth-order cross-correlation function of the surface displacement, defined by

$$N(y'_\alpha, \tau') = \overline{\zeta^2(y_\alpha, \tau)\, \zeta^2(y_\alpha + y'_\alpha, \tau + \tau')}. \quad (3.2)$$

In this, we have assumed that the surface wave field is both homogeneous and stationary, so that N depends only on the space and time separations.

From the definition of γ_w, it can be deduced that

$$\exp[i(\gamma_w^* - \gamma_w) x_3] = H\left(\frac{\omega^2}{c_w^2} - k_\alpha^2\right) + H\left(k_\alpha^2 - \frac{\omega^2}{c_w^2}\right) \exp\left(2\left(k_\alpha^2 - \frac{\omega^2}{c_w^2}\right)^{\frac{1}{2}} x_3\right),$$

H denoting the Heaviside unit function, so that (3.1) can be rewritten as

$$P_s(\boldsymbol{x},\omega) = \frac{\rho_w^4\,\omega^8}{4(2\pi)^2}\int_\infty N(y'_\alpha,\tau')\frac{|\gamma_a|^2}{|F_d(k_\alpha,\omega)|^2}\exp\left[\mathrm{i}(k_\alpha y'_\alpha+\omega\tau')\right] \times\left[H\left(\frac{\omega^2}{c_w^2}-k_\alpha^2\right)+H\left(k_\alpha^2-\frac{\omega^2}{c_w^2}\right)\exp\left(2\left(k_\alpha^2-\frac{\omega^2}{c_w^2}\right)^{\frac{1}{2}}x_3\right)\right]\mathrm{d}^2k_\alpha\,\mathrm{d}^2y'_\alpha\,\mathrm{d}\tau'.$$

Since we are concerned here with the sound field in deep water, this result can be simplified by letting $(-x_3)$ be very large. Then the second term in the square bracket is negligible and the power spectrum is independent of $\boldsymbol{x}$, that is,

$$P_s(\omega) = \frac{\rho_w^4\,\omega^8}{4(2\pi)^2}\int_\infty N(y'_\alpha,\tau')\frac{|\gamma_a|^2}{|F_d(k_\alpha,\omega)|^2}H\left(\frac{\omega^2}{c_w^2}-k_\alpha^2\right)\exp\left[\mathrm{i}(k_\alpha y'_\alpha+\omega\tau')\right]\mathrm{d}^2k_\alpha\,\mathrm{d}^2y'_\alpha\,\mathrm{d}\tau'. \tag{3.3}$$

For homogeneous and stationary ocean surface motions (both the wind fetch and duration being large), the surface displacement ζ can approximately be regarded as specified by a normal distribution (Phillips 1977). We further suppose that the joint distribution of ζ at two points is also normal. Then by following Batchelor's (1959) scheme, we find

$$N(y'_\alpha,\tau') = (\Pi(0,0))^2+2(\Pi(y'_\alpha,\tau'))^2, \tag{3.4}$$

where Π is the cross-correlation function of the surface wave field,

$$\Pi(y'_\alpha,\tau') = \overline{\zeta(y_\alpha,\tau)\,\zeta(y_\alpha+y'_\alpha,\tau+\tau')}. \tag{3.5}$$

The contribution to $P_s(\omega)$ from the first term on the right-hand side of (3.4) is zero unless k_α and ω vanish (which is not a consideration of our problem) so that (3.3) becomes

$$P_s(\omega) = \frac{\rho_w^4\,\omega^8}{2(2\pi)^2}\int_\infty \Pi^2(y'_\alpha,\tau')\frac{|\gamma_a|^2}{|F_d(k_\alpha,\omega)|^2}H\left(\frac{\omega^2}{c_w^2}-k_\alpha^2\right)\exp\left[\mathrm{i}(k_\alpha y'_\alpha+\omega\tau')\right]\mathrm{d}^2k_\alpha\,\mathrm{d}y'_\alpha\,\mathrm{d}\tau'.$$

Because of the Heaviside function, the integration with respect to k_α is actually restricted to the acoustic domain $|k_\alpha| < |\omega|/c_w$, in which we have

$$\frac{|\gamma_a|^2}{|F_d(k_\alpha,\omega)|^2} = \frac{1}{\rho_w^2\,\omega^4}\left(1+O\left(\frac{g^2}{\omega^2c_w^2}\right)+O\left(\frac{\rho_a}{\rho_w}\right)\right),$$

which gives

$$P_s(\omega) = \frac{\rho_w^2\,\omega^4}{2(2\pi)^2}\int_\infty \Pi^2(y'_\alpha,\tau')\,H\left(\frac{\omega^2}{c_w^2}-k_\alpha^2\right)\exp\left[\mathrm{i}(k_\alpha y'_\alpha+\omega\tau')\right]\mathrm{d}^2k_\alpha\,\mathrm{d}^2y'_\alpha\,\mathrm{d}\tau',$$

where terms smaller by $g^2/\omega^2c_w^2$ or ρ_a/ρ_w have been neglected. The k_α-integral can now be evaluated explicitly as

$$\int_\infty H\left(\frac{\omega^2}{c_w^2}-k_\alpha^2\right)\exp\left(\mathrm{i}k_\alpha y'_\alpha\right)\mathrm{d}^2k_\alpha = \frac{\pi\omega^2}{c_w^2}\left(J_0\left(\frac{\omega}{c_w}|y'_\alpha|\right)+J_2\left(\frac{\omega}{c_w}|y'_\alpha|\right)\right),$$

with J_0 and J_2 being, respectively, the zeroth- and second-order Bessel functions. From this it follows that

$$P_s(\omega) = \frac{\rho_w^2\,\omega^6}{8\pi c_w^2}\int_\infty \Pi^2(y'_\alpha,\tau')\left(J_0\left(\frac{\omega}{c_w}|y'_\alpha|\right)+J_2\left(\frac{\omega}{c_w}|y'_\alpha|\right)\right)e^{\mathrm{i}\omega\tau'}\,\mathrm{d}^2y'_\alpha\,\mathrm{d}\tau'.$$

In this integral the maximum value of y'_α can be replaced by the coherence length of the surface waves; Π^2 is negligible for large value of y'_α. Hence y'_α is at most equal to the surface wavelength so that $|y'_\alpha|\,\omega/c_w$ is of the same order as the ratio of the surface wavelength to the acoustic wavelength, which is usually very much less than unity. Thus the bracket in the integrand is effectively equal to one and the power spectrum becomes

$$P_s(\omega) = \frac{\rho_w^2\,\omega^6}{4(2\pi)^4 c_w^2}\hat{\Pi}(0,\omega) * \hat{\Pi}(0,\omega), \tag{3.6}$$

where the symbol $*$ denotes the three-dimensional convolution and $\hat{\Pi}(\eta_\alpha, \Omega)$ is the Fourier transform of (3.5) and is related to the mean square surface displacement by

$$\overline{\zeta^2} = \frac{1}{(2\pi)^3}\int_\infty \hat{\Pi}(\eta_\alpha, \Omega)\,\mathrm{d}^2\eta_\alpha\,\mathrm{d}\Omega.$$

Nonlinear surface motions generate a sound whose power spectral density $P_s(\omega)$ is independent of position. This sound is identical to the result obtained by Hughes (1976) who analysed this problem through the method of small perturbations in line with the Brekhovskikh scheme. This is the sound field that would be generated by weakly nonlinear interactions of ocean surface waves bounded above by a linearly disturbed atmosphere. This result does not seem to explicitly contain any singularities of the kind that lead to Olbers' paradox. This is because the natural surface wave speed (approximately equal to g/ω) is much smaller than the sound speed so that the singular peak at $F_d(k_\alpha, \omega) = 0$ of the spectrum (3.1) is confined to surface waves which decay exponentially with depth. However, (3.6) still implies a singular field because the surface wave field Π, driven by the turbulent airflow, is assumed both homogeneous and stationary in the entire physical space. This kind of surface motion can only be induced by a turbulent flow that itself is also homogeneous and stationary and of infinite extent. Such a problem is ill-posed; surface waves under an infinite region of turbulence have formally an infinite amplitude (which will be demonstrated in the next section). The sound field generated by this 'infinitely big' surface wave field will then also be infinite, or more precisely also ill-defined. To account for this problem correctly, it is necessary to assume that the turbulent sources have finite extent.

4. Surface waves induced by a finite region of turbulence

In linear theory, pressure fluctuations caused by the source distribution T_{ij} can be derived from the results of §2,

$$p(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int_\infty HT_{ij}(\boldsymbol{y},\tau)\,d_i\,d_j\frac{\rho_w\,\omega^2}{F_d(k_\alpha,\omega)}\exp\left(\frac{g}{2c_a^2}y_3 - \frac{g}{2c_w^2}x_3\right)$$
$$\times\, \mathrm{e}^{\mathrm{i}(\gamma_a y_3 - \gamma_w x_3)}\,\mathrm{e}^{-\mathrm{i}[k_\alpha(y_\alpha - x_\alpha)+\omega(\tau-t)]}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau\,\mathrm{d}^2k_\alpha\,\mathrm{d}\omega,$$

where $d_1 = -\mathrm{i}k_1$, $d_2 = -\mathrm{i}k_2$ and $d_3 = g/2c_a^2 + \mathrm{i}\gamma_a$. Alternatively, this pressure field can be expressed in the $(k_\alpha - \omega)$-space by the Fourier transform of p, which we denote by $\hat{p}$. To find the surface displacement, we first derive the pressure fluctuation on the mean water surface by letting x_3 vanish,

$$\hat{p}(-k_\alpha, 0, -\omega) = \int_\infty HT_{ij}(\boldsymbol{y},\tau)\,d_i\,d_j\frac{\rho_w\,\omega^2}{F_d(k_\alpha,\omega)}\exp\left(\frac{g}{2c_a^2}y_3\right)\mathrm{e}^{\mathrm{i}(\gamma_a y_3 - k_\alpha y_\alpha - \omega\tau)}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{4.1}$$

Because we are considering motions in the water that are essentially linear, the momentum equation in the vertical direction, $\rho_w \partial u_3/\partial t + \partial p/\partial x_3 + gp/c_w^2 = 0$, and the linearized boundary condition $u_3 = \partial\zeta/\partial t$ on $x_3 = 0$, can be used to relate p to the surface displacement ζ. It can be shown that

$$\hat{\zeta}(k_\alpha, \omega) = \frac{g/2c_w^2 - i\gamma_w}{\rho_w \omega^2}\hat{p}(k_\alpha, 0, \omega), \tag{4.2}$$

where $\hat{\zeta}$ is the Fourier transform of ζ. This result can be used to justify the scaling law $p_w' \sim \rho_w g\zeta$ used in §2. The pressure fluctuation (4.1) comes predominantly from gravity waves; the spectrum is peaked at $F_d(k_\alpha, \omega) = 0$, which gives approximately $\gamma_w = i\omega^2/g$. This immediately reduces (4.2) to $\hat{p} \sim \rho_w g\hat{\zeta}$, or equivalently $p_w' \sim \rho_w g\zeta$. Now on substituting (4.1) into (4.2) and taking the inverse Fourier transform, we find

$$\zeta(x_\alpha, t) = \int_\infty HT_{ij} G_{ij}\, e^{i\omega(t-\tau)}\, d^3\boldsymbol{y}\, d\tau\, d\omega, \tag{4.3}$$

where G_{ij} is given by

$$G_{ij} = \frac{1}{(2\pi)^3}\int_\infty d_i d_j \frac{g/2c_w^2 + i\gamma_w}{F_d(k_\alpha, \omega)} \exp\left(\frac{g}{2c_a^2} y_3\right) e^{i[\gamma_a y_3 - k_\alpha(y_\alpha - x_\alpha)]}\, d^2k_\alpha.$$

We calculate G_{ij} through the use of the residue theorem. The dominant contribution arises from the singularity $F_d(k_\alpha, \omega) = 0$, which gives rise to gravity waves. The contribution from the branch points $\gamma_a = 0$ and $\gamma_w = 0$ is negligible because it corresponds to sonic waves and decays more rapidly with distance from the source than gravity waves. From another point of view, those sonic surface waves, independent of gravity, can be neglected since we are seeking to determine the effects on sound generation of gravity waves. The position of the singularity can be determined by solving $F_d(k_\alpha, \omega) = 0$ iteratively in powers of ρ_a/ρ_w. The leading term shows that $g|k_\alpha| \approx \omega^2$, at which G_{ij} can be calculated in terms of Hankel functions which we express according to

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\left(-i[z - \tfrac{1}{2}\pi n - \tfrac{1}{4}\pi]\right). \tag{4.4}$$

This approximation is valid provided $|z|$ is large, which is the case of our problem for reasons that will shortly be apparent. The singularity at $g|k_\alpha| \approx \omega^2$ clearly demonstrates that the surface waves in our model are generated by a resonance mechanism as described by Phillips (1977). When the variations of the turbulence sources coincide with that of the natural surface waves, a wave field is excited. In our model the turbulence sources are essentially decoupled from the wave field. This necessarily implies that the surface waves are in a very early stage of their evolution so that the assumption of weak nonlinearity can be imposed. Hence we have

$$G_{ij} = \hat{y}_i \hat{y}_j \frac{i\omega^5}{2\pi\rho_w g^3}\left(\frac{\operatorname{sgn}(\omega)}{2\pi g|y_\alpha - x_\alpha|}\right)^{\frac{1}{2}} \exp\left[\frac{-\omega^2}{g} y_3 - i\left(\operatorname{sgn}(\omega)\frac{\omega^2}{g}|y_\alpha - x_\alpha| - \tfrac{1}{4}\pi\right)\right], \tag{4.5}$$

where $\operatorname{sgn}(\omega)$ denotes the sign function and $\hat{y}_i$ are directional factors defined by $\hat{y}_1 = y_1/|y_\alpha|$, $\hat{y}_2 = y_2/|y_\alpha|$ and $\hat{y}_3 = -i$. Now we multiply (4.3) by itself calculated at the space and time separations, x_α' and t', and take the Fourier transform with respect to them. This gives the three-dimensional surface wave spectrum

$$\hat{\Pi}(k_\alpha, \omega) = (2\pi)^2 \int_\infty \overline{HT_{ij} HT_{kl}'}\, G_{ij}(\omega)\, G_{kl}'(-\omega) \exp\left[i(k_\alpha x_\alpha' + \omega\tau')\right] d^3\boldsymbol{y}\, d^3\boldsymbol{y}'\, d\tau'\, dx_\alpha', \tag{4.6}$$

where G'_{kl} and T'_{kl} are calculated at $(y_\alpha + y'_\alpha, y'_3, \tau+\tau')$, and k_α and ω are now understood to be the surface wavenumber and frequency.

The exact calculation of (4.6) is very complicated and probably not necessary for our purpose. Our ultimate concern is the sound generated nonlinearly by surface waves. This kind of sound production is possible only when the surface source motions have supersonic phase velocities. In the weakly nonlinear theory, the phase velocity of the product $\exp\{i(\omega_1 t + k_1 x)\}$ and $\exp\{i(\omega_2 t + k_2 x)\}$ is equal to the sum of the frequencies divided by the sum of the wavenumbers of the two interacting surface waves, that is $(\omega_1+\omega_2)/(k_1+k_2)$. It exceeds the sound speed when the interacting gravity waves have nearly the same frequencies but travel in almost opposite directions; $\omega_1+\omega_2$ is then approximately twice the individual frequency and k_1+k_2 is very small. Hence we only need to know the surface waves that travel in opposite directions right below the turbulent airflow. It is this surface wave field that can interact to generate sound; those other waves that are far from their sources are all outgoing and provide very weak acoustic sources. This point considerably simplifies the calculations. Since interacting surface waves right below the turbulence are mostly generated by distant sources, it is reasonable to concentrate on the contribution from the integrated effects of those sources. This justifies the use of the approximate relation (4.4), since the turbulence source region is very much bigger, in dimension, than the surface wavelength, $|y_\alpha|\,\omega^2/g \gg 1$, and also, bearing this in mind, we can suppose that $|y_\alpha| \gg |x_\alpha|$ so that from (4.5)

$$G_{ij}(\omega)\, G'_{kl}(-\omega) = \hat{y}_i \hat{y}_j \hat{y}_k \hat{y}_l \frac{\omega^{10}}{(2\pi)^3 \rho_w^2 g^7 |y_\alpha|} \exp\left(-\frac{\omega^2}{g}(y_3+y'_3) + i\frac{\omega^2}{g}(y'_\alpha - x'_\alpha)\,\hat{y}_\alpha\right).$$

The surface wave spectrum (4.6) then becomes

$$\hat{\Pi}(k_\alpha, \omega) = \frac{\omega^{10}}{2\pi \rho_w^2 g^7} \int_\infty S_{ijkl} \frac{\hat{y}_i \hat{y}_j \hat{y}_k \hat{y}_l}{|y_\alpha|} \exp\left(-\frac{\omega^2}{g}(y_3+y'_3) + i\frac{\omega^2}{g}\hat{y}_\alpha y'_\alpha\right)$$
$$\times \exp\left(i\left(k_\alpha - \frac{\omega^2}{g}\hat{y}_\alpha\right) x'_\alpha\right) d^2x'_\alpha\, d^2y_\alpha\, d^2y'_\alpha\, dy_3\, dy'_3$$

where S_{ijkl} denotes the Fourier transform with respect to τ' of the source function $HT_{ij}\, HT'_{kl}$, namely,

$$S_{ijkl} = \int_\infty \overline{HT_{ij}\, HT'_{kl}}\, e^{i\omega\tau'}\, d\tau'.$$

It is now very clear that, because of the divergence of the y_α-integral, the surface wave field is infinite if the source region is of infinite extent. If we assume the distribution turbulence sources to be over a region of large but finite extent L, homogeneous and stationary over $|y_\alpha| < L$ and vanishing outside, the surface waves can be found by first calculating the x'_α-integral, which gives the result in terms of δ-functions, and then performing the y_α-integral in this finite region. This procedure leads to

$$\hat{\Pi}(k_\alpha, \omega) = \frac{4\pi L \omega^6}{\rho_w^2 g^5} X_{ijkl}\, b_i b_j b_k b_l\, \delta\left(\frac{g^2}{\omega^4}k_\alpha^2 - 1\right), \tag{4.7}$$

where $b_1 = (g/\omega^2)\,k_1$, $b_2 = (g/\omega^2)\,k_2$ and $b_3 = -i$, and X_{ijkl} is an integrated source function defined by

$$X_{ijkl} = \int_\infty S_{ijkl} \exp\left(\frac{-\omega^2}{g}(y_3+y'_3) + ik_\alpha y'_\alpha\right) dy_3\, dy'_3\, d^2y'_\alpha. \tag{4.8}$$

Equation (4.7) is the main result of this section. It shows that the surface wave spectrum, and hence the mean square surface displacement, is proportional to the linear dimension of the source region. With this we can unambiguously discuss the sound from surface wave interactions, which we do in the next section, after first estimating the r.m.s. surface displacement from (4.7) and comparing it with experiments to establish the relevance of our surface wave model. This is done by integrating (4.7) over k_α and ω, which yields the mean square surface elevation as

$$\overline{\zeta^2} = \frac{L}{2\pi\rho_w^2 g^7}\int_\infty \omega^{10} A_{ijkl} S_{ijkl} J_0\left(\frac{\omega^2}{g}|y'_\alpha|\right)\exp\left(\frac{-\omega^2}{g}(y_3+y'_3)\right)\mathrm{d}^2y'_\alpha\,\mathrm{d}y_3\,\mathrm{d}y'_3\,\mathrm{d}\omega, \quad (4.9)$$

where A_{ijkl} is a factor resulting from the k_α-integration and is of order one (for example, it is exactly equal to one when $i = j = k = l = 3$). We non-dimensionalize the integral by introducing, for the turbulence sources, the typical timescale t_0, frequency $\omega_0 = 2\pi/t_0$, lengthscale l and the vertical dimension of the source layer Δ. We also scale the source $HT_{ij}\,HT'_{kl}$ as $\rho_a^2 u^4$, u being the r.m.s. turbulence velocity. Hence we find

$$\overline{\zeta^2} \sim \left(\frac{\rho_a}{\rho_w}\right)^2 \frac{u^4 l^2 \Delta^2 L \omega_0^{10}}{g^7}. \quad (4.10)$$

In this result the timescales and lengthscales are those of turbulence sources. They can be related to parameters in the wave field. In an active wind-induced surface wave field, the dominant waves move at a phase speed nearly equal to the wind speed U, that is, their wave period is of the order $2\pi U/g$. In linear theory these waves must have been generated by those elements of the turbulence sources which have a matching timescale. Hence we have $t_0 \sim 2\pi U/g$. Consequently the lengthscale l is approximately $2\pi U^2/g$. The height of the source layer can be determined by examining (4.9). Because of the exponential factor in the integrand, only those sources adjacent to the water surface to within one wavelength have significant effects on surface wave production. The effective height Δ of the source layer can then be regarded as the lengthscale of the dominant waves, namely, $2\pi U^2 g$. With all these considerations, (4.10) simplifies to

$$\overline{\zeta^2} \sim (2\pi)^4 \left(\frac{\rho_a}{\rho_w}\right)^2 \frac{L u^4}{g U^2}.$$

In the turbulent airflow we can regard the r.m.s. velocity u as being of the same order as the friction velocity u_* so that $(u/U)^2 \approx (u_*/U)^2 \approx (1 \sim 3)\times 10^{-3}$ is actually the drag coefficient. Thus the r.m.s. surface displacement is

$$\zeta_{\mathrm{rms}} \sim 2.6\times 10^{-3}\left(\frac{Lu^2}{g}\right)^{\frac{1}{2}}. \quad (4.11)$$

This model is consistent with observations. A comparison can be made with the result summarized by Phillips who derived a formula from experiments,

$$\zeta_{\mathrm{rms}} \sim C\left(\frac{Lu_*^2}{g}\right)^{\frac{1}{2}}$$

(Phillips 1977) where C is a constant of the order $(8.7 \sim 12.6)\times 10^{-3}$. The r.m.s. turbulence velocity u in (4.11) is of the same order as u_*. It is then clear that (4.11) is close to Phillips' result.

5. Relative magnitude of the nonlinear interaction sound

In the Brekhovskikh theory, the atmosphere is supposed to be in a static state. But what is the effect and importance of higher-order terms in air? The neglect of the turbulent airflow may not be adequate because the aerial motions are not only the cause of surface waves, they also contain direct sound sources. This direct radiation may sometimes be the dominant sound. We examine in this section the relative magnitude of the noise from this aerial turbulence and the sound generated indirectly by surface waves. To do so, we calculate and compare the respective contributions to the sound power from unit area of turbulent sources and surface waves.

From the results of §2, the sound pressure due to turbulence, which we denote by p_t, can be derived from (2.10) as

$$p_t(\boldsymbol{x},t) = \frac{\mathrm{i}\rho_w}{(2\pi)^3}\int_\infty HT_{ij}\,d_i\,d_j\frac{1}{\rho_w\gamma_a+\rho_a\gamma_w}\,\mathrm{e}^{\mathrm{i}(\gamma_a y_3-\gamma_w x_3)} \times \mathrm{e}^{-\mathrm{i}[k_\alpha(y_\alpha-x_\alpha)+\omega(\tau-t)]}\,\mathrm{d}^2k_\alpha\,\mathrm{d}\omega\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \quad (5.1)$$

To calculate the sound power from a finite region of turbulence, it is sufficient to know the sound pressure far from the source region. This allows the use of the method of two-dimensional stationary phase (Jones 1982) to evaluate the k_α-integral, the result of which is the asymptotic solution of (5.1) as $|\boldsymbol{x}|\rightarrow\infty$,

$$p_t(\boldsymbol{x},t) = \frac{\hat{x}_i\,\hat{x}_j}{(2\pi)^2|\boldsymbol{x}|\,c_w^2}\,\frac{\rho_w\,x_3/|\boldsymbol{x}|}{\rho_w\,\hat{x}_3+\rho_a\,x_3/|\boldsymbol{x}|} \times\int_\infty \omega^2 HT_{ij}\exp\left(\mathrm{i}\frac{\omega}{c_w}(\hat{x}_m\,y_m-|\boldsymbol{x}|)+\mathrm{i}\omega(t-\tau)\right)\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau\,\mathrm{d}\omega,$$

where $\hat{x}_1 = x_1/|\boldsymbol{x}|$, $\hat{x}_2 = x_2/|\boldsymbol{x}|$ and $\hat{x}_3 = -(c_w^2/c_a^2-|x_\alpha|^2/|\boldsymbol{x}|^2)^{\frac{1}{2}}$. From this the mean square pressure can be found as

$$\overline{p_t^2} = \frac{\pi L^2}{(2\pi)^3\,|\boldsymbol{x}|^2\,c_w^4}\,\hat{x}_i\,\hat{x}_j\,\hat{x}_k\,\hat{x}_l\left(\frac{\rho_w\,x_3/|\boldsymbol{x}|}{\rho_w\,\hat{x}_3+\rho_a\,x_3/|\boldsymbol{x}|}\right)^2\int_\infty Y_{ijkl}\,\omega^4\,\mathrm{d}\omega, \quad (5.2)$$

where Y_{ijkl} is an integrated source function similar to X_{ijkl},

$$Y_{ijkl} = \int_\infty S_{ijkl}\exp\left[\mathrm{i}\frac{\omega}{c_w}\left(\hat{x}_3(y_3-y_3')-\hat{x}_\alpha\,y_\alpha'\right)\right]\mathrm{d}^2y_\alpha'\,\mathrm{d}y_3\,\mathrm{d}y_3'.$$

The sound power can now be calculated by dividing (5.2) by $\rho_w c_w$ and integrating it over a hemisphere centred at the origin. Furthermore, we divide the result by πL^2, the area of the turbulent source region, and drop the ω-integration. This gives the sound power from unit area of turbulence in unit frequency band, which is denoted by $W_t(\omega)$,

$$W_t(\omega) = \frac{\omega^4}{6\pi\rho_w\,c_w^3\,c_a^2}\int_\infty S_{ijkl}\,B_{ijkl}\exp\left(\mathrm{i}\frac{\omega}{c_a}(y_3'-y_3)\right)\mathrm{d}^2y_\alpha'\,\mathrm{d}y_3\,\mathrm{d}y_3', \quad (5.3)$$

with B_{ijkl} a factor of order one (similar to A_{ijkl} and equal to one if $i = j = k = l = 3$). We still let l be the correlation length in the source region and Δ the effective height of the turbulence source layer, and scale $\mathrm{d}^2y_\alpha'\,\mathrm{d}y_3\,\mathrm{d}y_3'$ as $l^2\Delta^2$ and S_{ijkl} as $\rho_a^2\,u^2(\omega)$, so that the integral in (5.3) is of the order $\rho_a^2\,l^2\Delta^2u^4(\omega)$, which leads to

$$W_t \sim \frac{\omega^4\rho_a^2}{6\pi\rho_w\,c_w^3\,c_a^2}\,l^2\Delta^2u^4(\omega). \quad (5.4)$$

Sound power radiated by surface-wave interactions can be similarly found from p_s, the sound pressure due to surface waves,

$$p_s(\boldsymbol{x},t) = \frac{-\rho_w^2}{2(2\pi)^3}\int_\infty \frac{\omega^2\gamma_a}{\rho_w\gamma_a+\rho_a\gamma_w}\zeta^2(y_\alpha,\tau)\,e^{-i[\gamma_w x_3+k_\alpha(y_\alpha-x_\alpha)+\omega(\tau-t)]}\,d^2k_\alpha\,d\omega\,d^2y_\alpha\,d\tau$$

$$= \frac{i\rho_w\hat{x}_3}{2(2\pi)^2|\boldsymbol{x}|c_w}\frac{\rho_w x_3/|\boldsymbol{x}|}{\rho_w\hat{x}_3+\rho_a x_3/|\boldsymbol{x}|}$$

$$\times\int_\infty \omega^3\zeta^2\exp\left(i\frac{\omega}{c_w}(\hat{x}_\alpha y_\alpha-|\boldsymbol{x}|)+i\omega(t-\tau)\right)d^2y_\alpha\,d\tau\,d\omega,$$

from which the mean square pressure is found to be

$$\overline{p_s^2} = \frac{\pi L^2\rho_w^2\hat{x}_3^2}{4(2\pi)^3|\boldsymbol{x}|^2c_w^2}\left(\frac{\rho_w x_3/|\boldsymbol{x}|}{\rho_w\hat{x}_3+\rho_a x_3/|\boldsymbol{x}|}\right)^2$$

$$\times\int_\infty \omega^6N(y'_\alpha,\tau')\exp\left(i\left[\omega\tau'-\frac{\omega}{c_w}\hat{x}_\alpha y'_\alpha\right]\right)d^2y'_\alpha\,d\tau'\,dy_3\,dy'_3,$$

where $N(y'_\alpha,\tau')$ is defined by (3.2) and we have taken the source region as πL^2, the same as that of the turbulent sources, because only in that region can surface waves interact. Following the same procedure as that in calculating $W_t(\omega)$ and making use of (3.4) for $N(y'_\alpha,\tau')$, we deduce that

$$W_s(\omega) = \frac{\rho_w\omega^6}{12\pi c_w^3}\int_\infty \Pi^2(y'_\alpha,\tau')J_0\left(\frac{\omega}{c_w}|y'_\alpha|\right)e^{i\omega\tau'}\,d^2y'_\alpha\,d\tau',$$

where $W_s(\omega)$ denotes the sound power from unit area of surface waves in unit frequency band and Π is defined by (3.5). This result can be rewritten in terms of $\hat{\Pi}(\eta_\alpha,\Omega)$, the surface wave spectrum, namely,

$$W_s(\omega) = \frac{\rho_w\omega^6}{12\pi c_w^3(2\pi)^5}\int_\infty \hat{\Pi}(\eta_\alpha,\Omega)\,\hat{\Pi}(\eta'_\alpha,w-\Omega)\,J_0\left(\frac{\omega}{c_w}|y'_\alpha|\right)$$

$$\times\exp\left[-i(\eta_\alpha+\eta'_\alpha)\,y'_\alpha\right]d^2y'_\alpha\,d^2\eta_\alpha\,d^2\eta'_\alpha\,d\Omega.$$

Again, since the source function in this result vanishes when $|y'_\alpha|$ exceeds a surface wavelength, the argument of the Bessel function J_0 is at most of the same order as the ratio of the surface wavelength to the acoustic wavelength, a factor usually very much less than one, so that the Bessel function can effectively be replaced by one and

$$W_s(\omega) = \frac{\rho_w\omega^6}{12\pi c_w^3(2\pi)^3}\int_\infty \hat{\Pi}(\eta_\alpha,\Omega)\,\hat{\Pi}(-\eta_\alpha,\omega-\Omega)\,d^2\eta_\alpha\,d\Omega, \tag{5.5}$$

For a finite region of turbulent airflow, we have derived the surface wave spectrum in the form of (4.7). The sound generated by these surface waves can then be calculated by directly substituting (4.7) into (5.5), which yields

$$W_s(\omega) = \frac{\omega^6L^2}{6\pi^2c_w^3\rho_w^3g^{10}}\int_\infty \Omega^6(\omega-\Omega)^6\,\delta\left(\frac{g^2}{\Omega^4}\eta_\alpha^2-1\right)\delta\left(\frac{g^2}{(\omega-\Omega)^4}\eta_\alpha^2-1\right)$$

$$\times X_{ijkl}\,b_i\,b_j\,b_k\,b_l\,X'_{mnpq}\,b_m\,b_n\,b_p\,b_q\,d^2\eta_\alpha\,d\Omega$$

where the dash implies that the function is evaluated at $(-\eta_\alpha,\omega-\Omega)$. The δ-functions can now be used to carry out the integrations, with the result

$$W_s(\omega) = \frac{L^2\omega^{23}}{12\times16^5\pi^2c_w^3\rho_w^3g^{12}}\int_0^{2\pi} X_{ijkl}\,b_i\,b_j\,b_k\,b_l\,X'_{mnpq}\,b_m\,b_n\,b_p\,b_q\,d\phi, \tag{5.6}$$

the integrand being calculated at $\eta_1 = (\omega^2/4g)\cos\phi$, $\eta_2 = (\omega^2/4g)\sin\phi$ and $\Omega = \frac{1}{2}\omega$. From the defining equation (4.8) of X_{ijkl}, it is apparent that the scaling of dy_3 and dy_3' is determined by the smaller value of the surface wavelength scale g/ω^2 and the effective height Δ of the turbulence source layer; if g/ω^2 is smaller the exponential factor in the integrand makes those sources more than a surface wavelength away from the air–water interface contribute a negligible part and hence $dy_3\,dy_3' \sim g^2/\omega^4$; on the other hand, if Δ is smaller, $dy_3\,dy_3'$ must be scaled as Δ^2 because the integrand S_{ijkl} is then itself negligible for sources more than Δ away from the surface. Considering all these, we find that the integral in (5.6) can be scaled as

$$\begin{cases} 2\pi[\rho_a^2\, u^4(\omega)\, l^2\Delta^2]^2 & \text{when } l < \dfrac{g}{\omega^2}, \\ 2\pi\left[\rho_a^2\, u^4(\omega)\, l^2\dfrac{g^2}{\omega^4}\right]^2 & \text{when } l < \dfrac{g}{\omega^2}. \end{cases}$$

Hence the surface wave generated sound power W_s becomes

$$W_s \sim \begin{cases} \dfrac{L^2\omega^{23}\rho_a^4\, l^4\Delta^4}{6\times 16^5\pi c_w^3\,\rho_w^3\, g^{12}}[u^4(\omega)]^2 & \text{when } l < \dfrac{g}{\omega^2}, \quad (5.7) \\ \dfrac{L^2\omega^{15}\rho_a^4\, l^4}{6\times 16^5\pi c_w^3\,\rho_w^3\, g^{8}}[u^4(\omega)]^2 & \text{when } l > \dfrac{g}{\omega^2}. \quad (5.8) \end{cases}$$

In (5.7) and (5.8), the high powers of ω seem to be a startling result. It becomes less surprising once it is recognized that W_s is related to the fourth power of the surface displacement, through (5.5), which itself is proportional to the square of the surface wavenumber because of the quadrupole property of the turbulence sources. In this way W_s has already been related to ω to the sixteenth power (the surface wavenumber scaling approximately on ω^2/g). It is not inconsistent with the governing equations. Actually it is well known that at low frequencies surface wave spectrum grows very rapidly to a peak as frequency increases (Phillips 1977). The low-frequency sound generated by such surface waves can also be expected to have a rapidly increasing low-frequency spectrum but the actual dependence of W_s on ω is almost impossible to check quantitatively because the dependence of $u^4(\omega)$, l and even possibly L on frequency is probably important but unknown. For the same reasons, the dependence of W_s on ω in the high-frequency region is probably never as high as the 'fifteenth power', as it seems to be in (5.8).

Having found the sound powers from unit area of turbulence and of surface waves respectively, we can now compare their relative importance. First we examine the case $l < g/\omega^2$, which corresponds to the low-frequency situation because the turbulence coherence scale l is of the order U/ω, and the condition $l < g/\omega^2$ then becomes $\omega < g/U$. Combining (5.4) and (5.7), it follows immediately that

$$\frac{W_t(\omega)}{W_s(\omega)} \sim 16^5\left(\frac{\rho_w}{\rho_a}\right)^2 \frac{g^{12}}{L^2 c_a^2\,\omega^{19} u^4(\omega)\, l^2\Delta^2}. \tag{5.9}$$

If we scale $l \sim \Delta$ on U/ω and write $u^4(\omega)$ in terms of the r.m.s. turbulence velocity u in the physical space, (5.9) is then equivalent to

$$\frac{W_t(\omega)}{W_s(\omega)} \sim 16^5\left(\frac{\rho_w}{\rho_a}\right)^2 \frac{g^{12}}{\omega^{14} L^2 c_a^2\, u^4 U^4}.$$

This result can be alternatively expressed in terms of the r.m.s. sound pressure,

$$\left(\frac{p_t}{p_s}\right)_{rms} \sim 4^5 \frac{\rho_w}{\rho_a} \frac{g^6}{\omega^7 L c_a u^2 U^2} \quad \text{when } \omega < \frac{g}{U}. \tag{5.10}$$

It is easy to show that this ratio is always bigger than one when $U > 1$ m/s and $L < 10^6$ m, which can be regarded as being the case in the real ocean. This implies that the aerial turbulence radiation is always dominant over the surface-induced low-frequency sound.

Now we consider the case $l > g/\omega$, or $\omega > g/U$. From (5.4) and (5.8), we have

$$\frac{W_t(\omega)}{W_s(\omega)} \sim 16^5 \left(\frac{\rho_w}{\rho_a}\right)^2 \frac{g^8}{L^2 c_a^2 \omega^{10} u^4} \left(\frac{\Delta}{l}\right)^2.$$

Again assuming $l \sim \Delta$, this becomes, in terms of the r.m.s. sound pressure,

$$\left(\frac{p_t}{p_s}\right)_{rms} \sim 4^5 \frac{\rho_w}{\rho_a} \frac{g^4}{L c_a \omega^5 u^2} \quad \text{when } \omega > \frac{g}{U}. \tag{5.11}$$

The results (5.10) and (5.11) reveal the relative acoustic radiation efficiency of the turbulence sources and surface waves. Their relative importance depends upon frequency, wind and the fetch of the turbulent airflow. At low frequency and small wind (the fetch correspondingly also small), the turbulent airflow is the dominant cause of underwater noise. In this case direct turbulence radiation overwhelms the surface-induced sound and weakly nonlinear interactions of surface waves play a negligible role. This confirms a conclusion discussed elsewhere (Guo 1987); sound and surface waves in this situation are essentially linearly related. It is their common source, the turbulent airflow, that gives their potential inter-connection; they are not cause and effect.

As frequency and/or wind speed increases, sound from surface wave interactions quickly becomes more important than that from the aerial turbulence. This is because high wind deforms the air–water interface more vigorously and nonlinearity becomes manifest. However the increase in wind speed limits also the applicability of any 'weakly nonlinear' theory. The Brekhovskikh theory suffers from two severe restrictions; it requires the ocean surface to be continuous and single-valued, and the surface wave slope to be much smaller than unity. These premises will inevitably be violated, as wind speed increases to vigorously agitate the air–water interface. High wind causes surface waves to depart significantly from being 'weakly nonlinear'. This becomes even clearer in terms of frequency. At high frequencies the relative magnitudes of turbulence sound and surface-induced sound are given by (5.11), from which it is apparent that the surface wave generated sound becomes appreciable, in comparison with the direct radiation from aerial turbulence, only in those high frequency regions where

$$\omega \gg 15.4 \left(\frac{g^4}{L c_a u^2}\right)^{\frac{1}{5}}.$$

The right-hand side can be re-arranged so that

$$\omega \gg 7.0 \left(\frac{L g u_*^2}{c_a^4}\right)^{\frac{1}{20}} \left[2.2 \frac{g}{u_*} \left(\frac{u_*^2}{L g}\right)^{\frac{1}{4}}\right], \tag{5.12}$$

where we have replaced the r.m.s. turbulence velocity by the friction velocity. It can be shown that the first factor in this inequality is always bigger than, but of the same order as, unity, in all practical situations, its minimum and maximum value being respectively 1.03 and 6.06 in the wind-speed range of 0.01–50 m/s and wind fetch range of 0.1–10^7 m, which obviously includes all possible circumstances in the natural ocean. Hence the condition (5.12) implies

$$\omega \gg 2.2 \frac{g}{u_*}\left(\frac{u_*^2}{Lg}\right)^{\frac{1}{4}}.$$

This is the condition that must be satisfied for the surface interaction sound to be dominant over the aerial turbulence radiation. But it has also been recognized as a critical condition in ocean wave studies, a condition under which ocean surface waves are in a fully nonlinear state where wave-breaking is the main feature of the surface motion (Phillips 1977). The physical mechanisms of ocean-sound production in this case are obviously different from both wave–wave interaction and pure turbulence radiation; processes such as splashing of water sprays on the ocean surface by breaking waves take over the leading role in generating noise.

This then leads us to conclude that the Brekhovskikh theory of ocean-sound generation by surface waves may not really be physically relevant; weakly nonlinear interactions between ocean waves are probably not a significant contributor to oceanic noise.

Now it is instructive to compare the problem discussed here with that in which the turbulent flow is over a flexible material surface that can support subsonic evanescent waves. In view of the similarity between the geometries of the two problems, there should be no difficulty in examining the material surface situation by following the method presented in this paper, and corresponding results can easily be obtained. However, since there will never be any breaking phenomena in that case and the surface motions there, if nonlinear at all, can always be regarded as 'weakly nonlinear', interactions of surface motions (and probably interactions between surface motions and turbulence) are then probably the dominant sources of high-frequency noise, provided the turbulence intensity is sufficiently strong to significantly deform the surface. Howe (1985) has suggested this possibility and roughly estimated some specific situations. Though it might be possible that nonlinear surface motions become important for some particular material surfaces, that is not so for the ocean-sound problem because the ocean surface would be bound to break! Fully nonlinear effects then provide a more efficient and dominant source of sound; weakly nonlinear interactions are not acoustically important in the ocean.

6. Conclusions

The mechanism of sound generation by weakly nonlinear interactions of ocean surface waves has been examined. The problem has been solved by making use of the appropriate form of Lighthill's acoustic analogy rather than the usual method of matched expansions in the Brekhovskikh scheme. This allows us to take account also of the effect of turbulent airflow. It has been shown that the aerial turbulence radiates a sound which overwhelms that of weakly nonlinear interacting surface waves at low frequency and small wind speed. We have shown that the wave–wave interaction mechanism becomes important only at high frequency and high wind. But it has been found that the condition for this mechanism to be appreciable, compared with the turbulent airflow, implies the precise condition at which fully nonlinear surface motions occur. Other more important physical mechanisms then take over the leading role. We then come to the conclusion that weakly nonlinear interactions of surface waves may not be a significant source of underwater sound; the Brekhovskikh theory is probably not relevant to the natural ocean.

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